The Siegel–Shidlovskii Theorem.

Hermite: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is transcendental.

Lindemann – Weierstrass

$\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$, $\mathbb{Z}$ lin. indep.

then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are alg. indep.

$\Rightarrow \log(2)$ transcendental.

$2\pi i$ " — "

$F \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ s.t. $F(e^{\alpha_1}, \dots, e^{\alpha_n}) = 0$

$\Rightarrow F = 0$.

<u>Easy</u>: $\exp(z) = \sum \frac{z^n}{n!} \in \overline{\mathbb{Q}}[[z]]$ is transcendental over $\overline{\mathbb{Q}}(z)$.

Pick $F(z, X) = \sum_{j=0}^{d} f_j(z) X^j$ $\quad f_j \in \overline{\mathbb{Q}}[z]$

$F(z, \exp(z)) = 0$. Choose $\deg f_0$ minimal

$\frac{\partial}{\partial z} F(z, \exp(z)) = 0$

$\hookrightarrow \sum_{j=0}^{d} \underbrace{(f_j'(z) + j f_j(z))}_{=0} \exp(z)^j$

$f_0' = 0$

$f_j = 0$ for $j > 0$

Show: $\alpha_1 \dots \alpha_n$ $\mathbb{Z}$-lin. independent

$\Rightarrow e^{\alpha_1 z}, \dots, e^{\alpha_n z} \in \overline{\mathbb{Q}}[[z]]$ are alg. indep.

Lindemann - Weierstrass: For every polyn. $P(X_1, \dots, X_n)$ is s.t.

$$P(e^{\alpha_1}, \dots, e^{\alpha_n}) = 0,$$

then there exists

$$Q[z, X_1, \dots X_n]$$

such that

1) $P = Q[1, X_1, \dots X_n]$

2) $Q[z, e^{\alpha_1 z}, \dots e^{\alpha_n z}] = 0.$

Siegel (1929) An E-function is power series

$$E(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \qquad a_n \in \overline{\mathbb{Q}}$$

such that

1) $LE = 0$ for some nonzero $L \in \overline{\mathbb{Q}}(z)[\partial]$

2) $|\sigma a_n| \leq C^n$ $\quad \forall n \geq 1$
$\forall \sigma \in \mathrm{Gal}_{\overline{\mathbb{Q}}}$

$$\mathrm{den}(a_1, \dots a_n) \leq C^n$$

$(1-z)e^z$

Beukers

Theorem Siegel - Shidlovskii (André, Beukers)
(Beukers: A refined version of ....)

Let $E_1, \dots, E_n$ be E-functions, satisfying

$$\begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix}' = \begin{pmatrix} a_{11}(z) & \cdots & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} \qquad a_{ij} \in \overline{\mathbb{Q}}(z)$$

Let $\alpha \in \overline{\mathbb{Q}}$ nonzero, not a pole of any a_{ij}.

Then for every $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ with

$$P(E_1(\alpha), \dots E_n(\alpha)) = 0$$

$\exists\, Q \in \overline{\mathbb{Q}}[z, X_1 \dots X_n]$ s.t.

$$P = Q(\alpha, X_1 \cdots X_n)$$

$$Q(z, E_1(z), \dots E_n(z)) = 0.$$

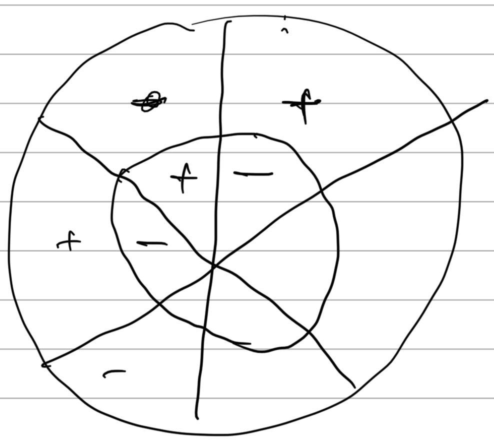
Examples of E-functions:

$e^{\alpha z}$ for any $\alpha \in \overline{\mathbb{Q}}$

Polynomials $\in \overline{\mathbb{Q}}[z]$

Sums, products, derivatives of E-fct.

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(z/2\right)^{2k+n}$$



J_n, J_m

distinct roots

Bourget's hypothesis.

$$E(z) = \int_0^1\int_0^1 e^{zxy}\,dx\,dy = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2\, n!} z^n$$